Linear systems – Final exam

Final exam 2021–2022, Tuesday 21 June 2022, 16:00 – 18:00

Problem 1

(4 + 4 + 8 + 14 = 30 points)

A power network with two (identical) generators can be modeled as

$$\begin{aligned} M\ddot{\theta}_1(t) + A\dot{\theta}_1(t) &= -B\sin\big(\theta_1(t) - \theta_2(t)\big) + u(t), \\ M\ddot{\theta}_2(t) + A\dot{\theta}_2(t) &= -B\sin\big(\theta_2(t) - \theta_1(t)\big) - P, \end{aligned}$$

where $\theta_1(t), \theta_2(t) \in \mathbb{R}$ are the Voltage angles of the generators. The generated power $u(t) \in \mathbb{R}$ is regarded as the input to the system, whereas P is the constant power demand. The positive constants M and A denote the inertias and damping constants for the generators, respectively, whereas $B > 0$ is related to the conductance of the line connecting the two generators.

(a) Define $x_1 = \dot{\theta}_1$, $x_2 = \dot{\theta}_2$, and $x_3 = \theta_1 - \theta_2$ where the former two can be interpreted as frequencies. Show that, in these new coordinates, the power network can be described as

$$\begin{aligned} M\dot{x}_1(t) &= -Ax_1(t) - B\sin x_3(t) + u(t), \\ M\dot{x}_2(t) &= -Ax_2(t) + B\sin x_3(t) - P, \\ \dot{x}_3(t) &= x_1(t) - x_2(t). \end{aligned} \tag{1}$$

To show this, consider x_1, for which we obtain

$$\begin{aligned} M\dot{x}_1 = M\ddot{\theta}_1 &= -A\dot{\theta}_1 - B\sin(\theta_1 - \theta_2) + u \\ &= -Ax_1 - B\sin(x_3) + u. \end{aligned} \tag{2}$$

Similarly, a direct computation gives

$$\begin{aligned} M\dot{x}_2 = M\ddot{\theta}_2 &= -A\dot{\theta}_2 + B\sin(\theta_1 - \theta_2) - P \\ &= -Ax_2 + B\sin(x_3) - P. \end{aligned} \tag{3}$$

Finally, considering x_3, we obtain

$$\dot{x}_3 = \dot{\theta}_1 - \dot{\theta}_2 = x_1 - x_2, \tag{4}$$

and thus the power network can be described as (1).

(b) Define the total energy

$$E(x_1, x_2, x_3) = \tfrac{1}{2}Mx_1^2 + \tfrac{1}{2}Mx_2^2 - B\cos x_3.$$

Take $u(t) = 0$ for all t and let $P = 0$. Show that the total energy in the power network is nonincreasing, i.e., $E(x_1(t), x_2(t), x_3(t))$ is a nonincreasing function of time.

We simply compute

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}E\big(x_1(t), x_2(t), x_3(t)\big) &= Mx_1\dot{x}_1 + Mx_2\dot{x}_2 + B\sin(x_3)\dot{x}_3 \\ &= -Ax_1^2 - B\sin(x_3)x_1 + ux_1 \\ &\quad -Ax_2^2 + B\sin(x_3)x_2 - Px_2 + B\sin(x_3)(x_1 - x_2) \\ &= -Ax_1^2 - Ax_2^2 \le 0, \end{aligned} \tag{5}$$

showing that $t \mapsto E(x_1(t), x_2(t), x_3(t))$ is nonincreasing.

(c) Let $P > 0$ be given and consider the constant input $u(t) = \bar{u}$. Show that an equilibrium point corresponding to $\bar{u}$ exists if and only if

$$-B \leq \frac{\bar{u}+P}{2} \leq B.$$

We have that, for given $P > 0$ and $\bar{u}$, that $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is an equilibrium point if

$$\begin{aligned} 0 &= -A\bar{x}_1 - B\sin\bar{x}_3 + \bar{u}, \\ 0 &= -A\bar{x}_2 + B\sin\bar{x}_3 - P, \\ 0 &= \bar{x}_1 - \bar{x}_2. \end{aligned} \tag{6}$$

Since $A > 0$, (6) holds if and only if

$$\begin{aligned} \bar{x}_1 &= \frac{1}{A}\left(-B\sin\bar{x}_3 + \bar{u}\right), \\ \bar{x}_2 &= \frac{1}{A}\left(B\sin\bar{x}_3 - P\right), \\ 0 &= \bar{x}_1 - \bar{x}_2. \end{aligned} \tag{7}$$

Substituting the $\bar{x}_1$ and $\bar{x}_2$ from the first two equations into the third yields

$$0 = \frac{1}{A}\left(-2B\sin\bar{x}_3 + \bar{u} + P\right), \tag{8}$$

Since $B > 0$, the latter holds if and only if

$$\sin\bar{x}_3 = \frac{\bar{u}+P}{2B}. \tag{9}$$

Consequently, (7) holds if and only if

$$\begin{aligned} \bar{x}_1 &= \frac{\bar{u}-P}{2A}, \\ \bar{x}_2 &= \frac{\bar{u}-P}{2A}, \\ \sin\bar{x}_3 &= \frac{\bar{u}+P}{2B}. \end{aligned} \tag{10}$$

We know that there exists θ such that $\sin\theta = c$ if and only if $-1 \leq c \leq 1$. Therefore, (10) has a solution if and only if

$$-1 \leq \frac{\bar{u}+P}{2B} \leq 1 \quad \Leftrightarrow \quad -B \leq \frac{\bar{u}+P}{2} \leq B. \tag{11}$$

As (10) holds if and only if (6) holds, and an equilibrium exists if and only if (6) holds, we conclude that an equilibrium corresponding to $\bar{u}$ exists if and only if

$$-B \leq \frac{\bar{u}+P}{2} \leq B. \tag{12}$$

(d) Let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ denote an equilibrium of (1) for the constant input $u(t) = \bar{u}$. Linearize the nonlinear system (1) around this equilibrium point.

Denote

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad f(x,u) = \begin{bmatrix} -\frac{A}{M}x_1 - \frac{B}{M}\sin x_3 + \frac{u}{M} \\ -\frac{A}{M}x_2 + \frac{B}{M}\sin x_3 - \frac{P}{M} \\ x_1 - x_2 \end{bmatrix}, \tag{13}$$

after which (1) can be written as

$$\dot{x}(t) = f\big(x(t), u(t)\big). \tag{14}$$

Given the equilibrium

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}, \tag{15}$$

corresponding to $\bar{u}$, define the perturbations

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}, \tag{16}$$

for which the dynamics can be approximated by the linear system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t). \tag{17}$$

Here, $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}), \qquad \tilde{B} = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}). \tag{18}$$

We compute

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} -\frac{A}{M} & 0 & -\frac{B}{M}\cos x_3 \\ 0 & -\frac{A}{M} & \frac{B}{M}\cos x_3 \\ 1 & -1 & 0 \end{bmatrix}, \tag{19}$$

such that

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} -\frac{A}{M} & 0 & -\frac{B}{M}\cos \bar{x}_3 \\ 0 & -\frac{A}{M} & \frac{B}{M}\cos \bar{x}_3 \\ 1 & -1 & 0 \end{bmatrix}, \tag{20}$$

Similarly,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{1}{M} \\ 0 \\ 0 \end{bmatrix}. \tag{21}$$

Problem 2 (5 + 5 + 10 + 8 + 6 = 34 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and with

$$A = \begin{bmatrix} 8 & 5 \\ -12 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

(a) Is the system internally stable?

To verify stability, consider the characteristic polynomial of A given as

$$\Delta_A(s) = \det(sI - A) = \begin{vmatrix} s-8 & -5 \\ 12 & s+7 \end{vmatrix} = (s-8)(s+7) + 60 = s^2 - s + 4. \tag{22}$$

We know that a polynomial of degree 2 is stable (and, hence, the system is internally stable) if and only if all coefficients are nonzero and have the same sign. Clearly, this means that the system is not internally stable.

Alternatively, we may explicitly compute the eigenvalues by solving $\Delta_A(\lambda) = 0$, leading to

$$\lambda_1 = \frac{1 + i\sqrt{15}}{2}, \qquad \lambda_2 = \frac{1 - i\sqrt{15}}{2}, \tag{23}$$

with i the imaginary unit. As both eigenvalues have a positive real part, the system is not internally stable.

(b) Verify that the system is controllable.

By direct computation,

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & 5 \end{bmatrix}. \tag{24}$$

It is readily seen that

$$\operatorname{rank} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & -3 \\ 1 & 5 \end{bmatrix} = 2, \tag{25}$$

such that the system is controllable.

(c) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From the computation of the characteristic polynomial in (22) we know that

$$\Delta_A(s) = s^2 + a_1 s + a_0, \tag{26}$$

with $a_1 = -1$ and $a_0 = 4$. As the matrix pair (A, B) is controllable by (b), there exists a nonsingular matrix T that transforms the system to the controllability canonical form, i.e.,

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{27}$$

From this, we immediately conclude that

$$\alpha_1 = -a_0 = -4, \qquad \alpha_2 = -a_1 = 1. \tag{28}$$

To find the corresponding matrix T, define

$$q_2 = B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{29}$$

and

$$q_1 = AB + a_1 B = \begin{bmatrix} -3 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}. \tag{30}$$

Now, define

$$T^{-1} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 4 & 1 \end{bmatrix}, \tag{31}$$

which is the desired transformation matrix. Now, we have

$$T = \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ -4 & -2 \end{bmatrix}. \tag{32}$$

(d) Use the matrix T from (c) to design a state feedback controller $u(t) = Fx(t)$ such that the resulting closed-loop system satisfies $\sigma(A + BF) = \{-2, -3\}$.

As a first step, note that, for any nonsingular T,

$$\sigma\big(T(A + BF)T^{-1}\big) = \sigma(A + BF). \tag{33}$$

Define $\bar{F} = FT^{-1}$ such that

$$T(A + BF)T^{-1} = TAT^{-1} + TB\bar{F}. \tag{34}$$

Using the matrix T that achieves (27) and after denoting

$$\bar{F} = \begin{bmatrix} f_0 & f_1 \end{bmatrix}, \tag{35}$$

we obtain

$$TAT^{-1} + TB\bar{F} = \begin{bmatrix} 0 & 1 \\ f_0 - a_0 & f_1 - a_1 \end{bmatrix}. \tag{36}$$

As this matrix is in companion form, we can find its characteristic polynomial as

$$\Delta_{T(A+BF)T^{-1}}(s) = s^2 + (a_1 - f_1)s + (a_0 - f_0). \tag{37}$$

Given the desired eigenvalue locations, we obtain the desired characteristic polynomial

$$p(s) = (s + 2)(s + 3) = s^2 + 5s + 6. \tag{38}$$

Comparing (37) with (38) leads to

$$\left.\begin{aligned} a_1 - f_1 &= 5 \\ a_0 - f_0 &= 6 \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} f_1 &= a_1 - 5 = -1 - 5 = -6 \\ f_0 &= a_0 - 6 = 4 - 6 \quad = -2 \end{aligned}\right. \tag{39}$$

such that

$$\bar{F} = \begin{bmatrix} f_0 & f_1 \end{bmatrix} = \begin{bmatrix} -2 & -6 \end{bmatrix}. \tag{40}$$

Finally,

$$F = \bar{F}T = \tfrac{1}{2} \begin{bmatrix} -2 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 11 & 5 \end{bmatrix}. \tag{41}$$

(e) Now, let $u(t) = Fx(t) + v(t)$ with $v(t) \in \mathbb{R}$ a new "virtual" input, leading to the system

$$\dot{x}(t) = (A + BF)x(t) + Bv(t), \qquad y(t) = Cx(t).$$

To determine the transfer function, first compute

$$A_F = A + BF = \begin{bmatrix} 8 & 5 \\ -12 & -7 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 11 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -1 & -2 \end{bmatrix}, \tag{42}$$

such that, using Cramer's rule,

$$(sI - A_F)^{-1} = \begin{bmatrix} s+3 & 0 \\ 1 & s+2 \end{bmatrix} = \frac{1}{(s+3)(s+2)} \begin{bmatrix} s+2 & 0 \\ -1 & s+3 \end{bmatrix}. \tag{43}$$

Now, the transfer function can be obtained as

$$T(s) = C(sI - A_F)^{-1}B = \frac{1}{(s+3)(s+2)} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ -1 & s+3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{s+6}{(s+3)(s+2)}. \tag{44}$$

Note that the poles of this transfer function are -3 and -2, which are exactly the desired closed-loop eigenvalues of problem (d). We could say that we have placed the poles at those desired locations, which explains the terminology pole placement.

Problem 3 (4 + 6 + 16 = 26 points)

Consider the linear system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t)$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. Denote by $x(t; x_0) = e^{At}x_0$ the state trajectory of the system for initial condition $x(0) = x_0$. Define

$$\mathcal{S} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \lim_{t\to\infty} x(t; x_0) = 0 \right\}.$$

(a) We know that $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ if and only if $\mathcal{S}$ is nonempty and the implication

$$x_1, x_2 \in \mathcal{S}, \quad \alpha_1, \alpha_2 \in \mathbb{R} \quad \Longrightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{S} \tag{45}$$

holds. To show this, note that $0 \in \mathcal{S}$, hence $\mathcal{S}$ is nonempty. Let $x_1, x_2 \in \mathcal{S}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then,

$$\begin{aligned} x\big(t; \alpha_1 x_1 + \alpha_2 x_2\big) &= e^{At}\big(\alpha_1 x_1 + \alpha_2 x_2\big) \\ &= \alpha_1 e^{At} x_1 + \alpha_2 e^{At} x_2 = \alpha_1 x(t; x_1) + \alpha_2 x(t; x_2). \end{aligned} \tag{46}$$

As we have that

$$\lim_{t\to\infty} x(t; x_i) = 0 \tag{47}$$

for $i = 1, 2$ (since $x_1, x_2 \in \mathcal{S}$), this implies

$$\lim_{t\to\infty} x\big(t; \alpha_1 x_1 + \alpha_2 x_2\big) = 0, \tag{48}$$

i.e., $\alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{S}$ as desired.

(b) Recall that $\mathcal{S}$ is A-invariant if the implication

$$x_0 \in \mathcal{S} \quad \Longrightarrow \quad Ax_0 \in \mathcal{S} \tag{49}$$

holds. Let $x_0 \in \mathcal{S}$ and consider Ax_0. Then,

$$x\big(t; Ax_0\big) = e^{At}Ax_0 = Ae^{At}x_0 = Ax(t; x_0), \tag{50}$$

where we have used the fact that A and e^{At} commute for each $t \in \mathbb{R}$. However, we know that $\lim_{t\to\infty} x(t; x_0) = 0$, which implies

$$\lim_{t\to\infty} x(t; Ax_0) = 0, \tag{51}$$

such that $\mathcal{S}$ is A-invariant.

(c) We are asked to show that the system is detectable if

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \subset \mathcal{S}. \tag{52}$$

By the Hautus test, we know that the system is detectable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \tag{53}$$

for all $\lambda \in \sigma(A)$ with $\mathrm{Re}(\lambda) \geq 0$. It is thus sufficient to show that (52) implies (53). Let $\lambda \in \sigma(A)$ with $\mathrm{Re}(\lambda) \geq 0$ and let $v \in \mathbb{C}^n$ satisfy

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0. \tag{54}$$

This can be written as

$$Av = \lambda v, \qquad Cv = 0. \tag{55}$$

Note that we also have that $A^k v = \lambda^k v$ for all $k = 0, 1, \ldots$, leading to $CA^k v = \lambda^k Cv = 0$. In particular, this means

$$v \in \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{56}$$

By (52), this implies that $v \in \mathcal{S}$. Hence,

$$\lim_{t \to \infty} x(t; v) = 0 \tag{57}$$

But, we also know that

$$x(t; v) = e^{At} v = \left(I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \cdots \right) v = \left(I + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \cdots \right) v = e^{\lambda t} v. \tag{58}$$

By (57) and the assumption that $\mathrm{Re}(\sigma) \geq 0$, this can only hold if $v = 0$. Hence, a $v \in \mathbb{C}^n$ satisfying (54) is necessarily the zero vector. This implies (53), which is the desired result.

(10 points free)